GALVIN–MYCIELSKI–SOLOVAY THEOREM

AND ITS PITFALLS

- Prikry and GMS Theorems
- Extension to locally compact groups
- Failure for non-locally compact groups

THM (Galvin, Mycielski, Solovay 1976)

A set $X \subseteq \mathbb{R}$ is Smz if and only if

$X + M \neq \mathbb{R}$ for each meager set $M \subseteq \mathbb{R}$.

$\hookrightarrow \quad = \{x + m : x \in X, m \in M\}$

Samsonite

POLISH GROUPS

- Polish group : topological group
  separable, completely metrizable

- Every Polish group
  - → has a complete metric
  - → has a left-invariant metric
  - → but some may have no metric that is both complete and left-invariant
  - → every (two-sided) invariant metric is complete
  - → CLI group : has a complete left-invariant metric
  - → TSI group : has an invariant
    - → every abelian or compact is TSI

SMZ IN POLISH GROUPS

DEF Let $\mathbb{G}$ be a Polish group

$X \subseteq \mathbb{G}$ has (left) strong measure zero if

→ it is SMZ w.r.t. (any) left-invariant metric

→ equivalently, if for any sequence $\langle U_n \rangle$ of open sets (equiv. nbhds of 1) there is a sequence $\langle x_n \rangle$ in $\mathbb{G}$ s.t. $X \subseteq \bigcup_n x_n U_n$

→ $\mathrm{SMZ}(\mathbb{G})$ is bi-invariant

→ but if X is (left) SMZ, X^{-1} does not have to:

Hrušák, Zapletal 2016: (cov $\mathcal{M} = \mathfrak{c}$) $\exists X \subseteq S_\infty$ ∘ SMZ ∘ X^{-1} not SMZ

→ but if $\mathbb{G}$ is σ-compact, then X SMZ $\Leftrightarrow$ X^{-1} SMZ

PRIKRY'S THEOREM

THM (Prikry 1970) Let $X \subseteq G$.
If $XM \neq G$ for each M meager, then X is Smz.

Proof

- Let $Q \subseteq G$ be countable dense, g gauge
- $\exists\, U \supseteq Q$ G_δ-set such that $\mathcal{H}^g(U) = 0$
- $M := G \setminus U^{-1}$ is meager
- $\exists\, z \notin XM$ .... $X \subseteq zU$
- $\mathcal{H}^g(X) \leq \mathcal{H}^g(zU) = \mathcal{H}^g(U) = 0$
  ↳ left invariance

CONVERSE TO PRIKRY'S THM

Group-free version

THM (Hrušák, Wohofsky, Zindulka 2016)

Let
- T be a σ-compact metric space
- X be a seperable locally cpt metric space
- $\phi : T \times X \to X$ continuous

  $\forall y \in T \ \dots \ \phi(\{y\} \times P)$ nowhere dense whenever $P \subseteq X$ is nowhere dense

If $S \subseteq T$ is Smz and $M \subseteq X$ is meager, then $\phi(S \times M) \neq X$.

$Y = G$

COR 2

Let $\phi : G \curvearrowright X$ be an action of a σ-compact Polish group on a σ-compact space X.
If $S \subseteq G$ is Smz and $M \subseteq X$ meager, then $\phi(S \times M) \neq X$.

$X = Y = G$, $\phi(x,y) = xy$ (or $x^{-1}y$, yx ...)

COR 3

(Kysiak 2000, Fremlin 2008)

Let G be a σ-compact Polish group.
If $S \subseteq G$ is Smz and $M \subseteq G$ meager, then

- $SM \neq G$
- $MS \neq G$
- S^{-1} is Smz ...

LEMMA (Simplified version)

Let
- K, X be compact metric spaces
- $P \subseteq X$ compact nwd
- $\phi : K \times X \to X$ ... $\forall y \in K \quad \phi(\{y\} \times P)$ nwd

Then

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \; \forall y \in K \; \exists z \in X \quad B(z, \delta) \subseteq B(z, \varepsilon) \setminus \phi(B(y, \delta) \times P)$$

Proof ?

$\Rightarrow \forall \varepsilon \; \forall x \; \forall y \; \exists \delta' \; \exists z \quad B(z, \delta') \subseteq B(z, \varepsilon) \setminus \phi(\{y\} \times P)$

$\Rightarrow \forall \varepsilon \; \forall x \; \forall y \; \exists \delta \; \exists z \quad B(z, \delta) \subseteq B(z, \varepsilon) \setminus \phi(B(y, \delta) \times P)$

deflate $\qquad$ inflate

$\Rightarrow$ compactness:

switch quantifiers

PROOF OF THE THEOREM

Simplified version:

- Y, X compact, $\phi : Y \times X \to X$
- $P \subseteq X$ nwd, $S \subseteq Y$ Smz
- want: $\phi(S \times P) \neq X$

- recursion: $\langle \varepsilon_n \rangle$ such that

$$\forall x \, \forall y \, \exists z \quad B(z, \varepsilon_{n+1}) \subseteq B(x, \varepsilon_n) \setminus \phi(B(y, \varepsilon_{n+1}) \times P)$$

- S is Smz: $\exists \{E_n\}$ $\langle \varepsilon_n \rangle$-fine cover of S

  $\hookrightarrow \subseteq B(x_n, \varepsilon_n)$

  $$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \setminus \phi(E_{n+1} \times P)$$

- $\bigcap_n B(x_n, \varepsilon_n) = \{z\} \ldots$

  $$\ldots \; z \notin \bigcup_n \phi(E_n \times P) \supseteq \phi(S \times P)$$

UNIFORMLY MEAGER SETS

DEF (HWZ 2016)

→ $A \subseteq X$ is uniformly nowhere dense if

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \exists y \in X \ B(y, \delta) \subseteq B(x, \varepsilon) \setminus A$

→ $A \subseteq X$ is uniformly meager if it is a countable union of uniformly nwd sets

→ A σ-compact meager set is uniformly meager

→ The proof works for unif. meager

THM (HWZ 2016) Let G be a TSI Polish group. If $S \subseteq G$ is SMZ, then

$SM \neq G$ for each uniformly meager M

- If X is locally compact then
  meager = uniformly meager
- If X is nowhere locally compact then
  meager $\neq$ uniformly meager
- Polish group is locally compact $\Longleftrightarrow$
  meager = uniformly meager

<u>PROP</u> (HWZ2016)

(CH) There is $X \subseteq \mathbb{Z}^\omega$ such that

$\longrightarrow$ $X + M \neq \mathbb{Z}^\omega$ for all unif. meager M

$\longrightarrow$ X is not SmZ

NON - LOCALLY COMPACT CASE

- There are several proofs:
  - recursion over ω
  - recursion over $\omega^{<\omega}$
  - from the Galvin's game

  All of them require compactness
- meager $\rightsquigarrow$ uniformly meager fails
- Borel Conjecture: GMS Thm holds for all groups trivially
- What about CH?

ANTI - GMS SET

DEF (HWZ 2016)

A nonempty, closed set $M \subseteq \mathbb{G}$ is antiGMS if

- M is nowhere dense
- $\forall \langle \varepsilon_n \rangle \ \exists \{E_n\}$ $\langle \varepsilon_n \rangle$ - fine such that

  $\forall z \in \mathbb{G} \quad z \cdot (\bigcup_n E_n) \cap M$ is dense in M

THM (HWZ 2016)

(CH) If $\mathbb{G}$ contains an antiGMS set M, then $\exists S \subseteq \mathbb{G}$ $S \sim z$ such that $SM^{-1} = \mathbb{G}$.

- So GMS Theorem fails for $\mathbb{G}$.

Proof

- $\mathbb{G} = \{x_\alpha : \alpha < \omega_1\}$ enumerates $\mathbb{G}$
- $\{\langle \varepsilon_n^\alpha \rangle : \alpha < \omega_1\}$ enumerates $(0,\infty)^\omega$
- $\alpha < \omega_1 \mapsto \langle z_n^\alpha : n \in \omega \rangle$ $\forall z$ $z \cdot \bigcup_n B(z_n^\alpha, \varepsilon_n^\alpha) \cap M$ comeager in M

$$U_\alpha = \bigcup_n B(z_n^\alpha, \varepsilon_n^\alpha)$$

- Baire Category Thm: $\forall \alpha \; \exists m_\alpha \in M \cap x_\alpha^{-1} \cdot \bigcap_{\beta \leq \alpha} U_\beta$

$\Rightarrow$ $y_\alpha := x_\alpha m_\alpha^{-1} \in \bigcap_{\beta \leq \alpha} U_\beta$

- $S := \{y_\alpha : \alpha < \omega_1\}$

$\Rightarrow$ $x_\alpha = y_\alpha m_\alpha$, hence $SM^{-1} = \mathbb{G}$

$\Rightarrow$ Let $\langle \varepsilon_n \rangle \in (0,\infty)^\omega$. $\langle \varepsilon_{2n} \rangle = \langle \varepsilon_n^\alpha \rangle$ $\mapsto$ listed

$\Rightarrow$ $\forall \beta \geq \alpha$ $y_\beta \in U_\alpha$

$\Rightarrow$ $X \setminus U_\alpha \subseteq \{y_\beta : \beta < \alpha\}$ countable

$\Rightarrow$ $S \subseteq \underbrace{\{y_\beta : \beta < \alpha\}}_{\subseteq \bigcup_n B(t_n, \varepsilon_{2n+1})} \cup \underbrace{U_\alpha}_{\subseteq \bigcup_n B(x_n^\alpha, \varepsilon_{2n})}$

THM (HWZ 2016)

$\mathbb{Z}^\omega$ contains an antiGMS set.

- enumerate $\mathbb{Z}^{<\omega} = \{s_n : n \in \omega\}$
- $\langle \sigma_n : n \in \omega \rangle$ such that
  - $s_n \subseteq \sigma_n$
  - $\forall m$ at most one $\sigma_n \in \mathbb{Z}^m$
- $G = \bigcup_n \langle \sigma_n \rangle$ is open, dense but sparse
- $M = \mathbb{Z}^\omega \setminus G$ closed nwd
- M is antiGMS

CoRo (CH) GMS Theorem fails for $\mathbb{Z}^\omega$

SUBMEASURES

DEF $\mu : \mathcal{P}(G) \longrightarrow [0,\infty]$ is a **submeasure** if it is

(G → Polish group)

- σ-subadditive
- outer-regular: $\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ open}\}$
- non-atomic: $\forall x \in G \quad \mu(\{x\}) = 0$
- left-invariant: $\mu(xA) = \mu(A)$

→ If G is σ-compact: Haar measure is a submeasure

THM (Hrušák, Zapletal 2016)
There is a nontrivial submeasure on every Polish group.

THM (Grebík) Let $S \subseteq G$, TFAE (G → Polish group)

(i) S is SMZ

(ii) $\mu(S) = 0$ for each submeasure on G.

THM (Hrušák, Zapletal 2016)
Every TSI, not locally compact Polish group contains an antiGMS set.

COROLLARY (CH) If G is a TSI Polish group, then TFAE
(i) GMS Theorem holds for G
(ii) G is locally compact

THM (Hrušák, Zapletal 2016)
Every non-locally-compact closed subgroup of S_∞ contains an antiGMS set.

Questions:
→ Does the dichotomy extend beyond TSI?
→ What should replace "meager" to get GSM Theorem for all Polish groups?

SHARP MEASURE ZERO

- The notion
- Hausdorff measures
- Borel-like definition
- Products
- Galvin-like game
- Partition arrow

2^ω vs $\mathbb{R}$

- metric on 2^ω: $d(x,y) = 2^{-|x \wedge y|}$
- $T: \begin{cases} 2^\omega \to [0,1] \\ x \mapsto \frac{1}{2} \sum 2^{-n} x(n) \end{cases}$ "canonical map"
- T is Lipschitz
- Tsaban, Weiss 2005:

$\forall E \subseteq [0,1] \ \exists \ E_0 \cup E_1 = T^{-1}(E)$ .... $\operatorname{diam} E_i \leq \operatorname{diam} E$

- $X \subseteq 2^\omega$ is Sur $\iff$ $T(X) \subseteq \mathbb{R}$ is Sur

$$\begin{array}{ccc} \Updownarrow \text{GMS} & & \Updownarrow \\ X + M \neq 2^\omega & \iff & X + M \neq \mathbb{R} \end{array}$$

$\rightarrow$ $X \subseteq \left\{ \begin{matrix} 2^\omega \\ \mathbb{R} \end{matrix} \right\}$ is meager-additive if

$\forall M$ meager $X + M$ is meager

<u>QUESTION</u> : Is it true that

$X \subseteq 2^\omega$ is meager-additive $\Leftrightarrow T(X) \subseteq \mathbb{R}$ is meager-add?

$\rightarrow$ We need GMS Thm for "meager-additive".

- Y, X compact, $\phi : Y \times X \to X$
- $P \subseteq X$ nwd, $S \subseteq Y$ Smz
- want: $\phi(S \times P) \neq X$

- recursion: $\langle \varepsilon_n \rangle$ such that

$$\forall x \, \forall y \, \exists z \quad B(z, \varepsilon_{n+1}) \subseteq B(x, \varepsilon_n) \setminus \phi(B(y, \varepsilon_{n+1}) \times P)$$

- S is Smz: $\exists \{E_n\}$ $\langle \varepsilon_n \rangle$-fine cover of S

$\hookrightarrow \subseteq B(x_n, \varepsilon_n)$

$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \setminus \phi(E_{n+1} \times P)$$

- $\bigcap_n B(x_n, \varepsilon_n) = \{z\}$ ...

$$\ldots \; z \notin \bigcup_n \phi(E_n \times P) \supseteq \phi(S \times P)$$

SHARP MEASURE ZERO

DEF A set $S \subseteq X$ in a complete metric space X has **SHARP MEASURE ZERO** if, for each gauge g there is a σ-compact set K, $S \subseteq K$, $\mathcal{H}^g(K) = 0$.

UPPER HAUSDORFF MEASURE

$\longrightarrow$ $\overline{\mathcal{H}}^g_\delta(E) = \inf\left\{ \sum_n g(\operatorname{diam} E_n) : \{E_n\} \text{ is a } \underline{\text{finite}} \text{ cover of } E \right\}$

$\longrightarrow$ $\overline{\mathcal{H}}^g_0(E) = \sup_{\delta > 0} \mathcal{H}^g_\delta(E)$

$\hookrightarrow$ finitely subadditive but not a measure

$\longrightarrow$ $\overline{\mathcal{H}}^g(E) = \inf\left\{ \sum \overline{\mathcal{H}}^g_0(E_n) : \{E_n\} \text{ a countable cover of } E \right\}$

$\hookrightarrow$ measure

$\longrightarrow$ X complete : $\overline{\mathcal{H}}^g(E) = \inf\left\{ \mathcal{H}^g(K) : K \supseteq E \ \sigma\text{-compact} \right\}$

$\longrightarrow$ $\overline{\mathcal{H}}^g(E) < \infty \Rightarrow E$ σ-totally bounded

SHARP MEASURE ZERO — AGAIN

DEF X has **SHARP MEASURE ZERO** ($Smz^{\#}$) if $\overline{\mathcal{H}}^{g}(X) = 0$ for every gauge g.

→ $Smz^{\#}(X)$ – all $Smz^{\#}$ subsets of X

→ σ-ideal

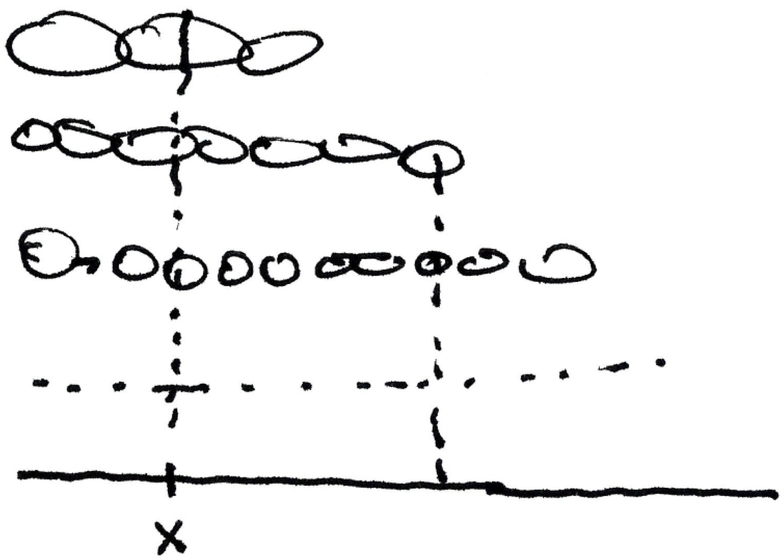
→ preserved by uniformly continuous maps

→ but if X is σ-compact or complete, then $S \subseteq X$ $Smz^{\#} \Rightarrow f(S)$ $Smz^{\#}$ for each continuous map f on X.

→ X $Smz^{\#} \Rightarrow$ X σ-totally bounded

BOREL-LIKE DEFINITION

RECALL : $\mathcal{H}^g(X)=0 \dots \exists\ \lambda$-cover $\sum_n g(\operatorname{diam} E_n)<\infty$



Each x is covered by almost all levels.

→ $\{G_n\}$ is a γ-cover of X if $\forall x\ \forall_n^\infty\ x\in G_n$

→ $\{E_n\}$ is a γ-groupable cover if there is a partition $\omega = I_0 \cup I_1 \cup \dots$ into finite sets s.t. $\{\bigcup_{n\in I_k} E_n : k\in\omega\}$ is a γ-cover.

LEMMA $E\in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^g) \iff E$ has a γ-groupable cover s.t. $\sum_n g(\operatorname{diam} E_n)<\infty$

(almost $\overline{\mathcal{H}}^g(E)=0$)

BOREL-LIKE DEFINITION

$\mathcal{N}_\sigma(\overline{\mathcal{H}}_0^g)$ .... σ-ideal generated by the sets with $\overline{\mathcal{H}}_0^g(E) = 0$.

LEMMA $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^g) \iff$

$\iff \exists$ γ-groupable cover $\{E_n\}$ s.t. $\sum_n g(dE_n) < \infty$

THM (Oz 2019) TFAE

(i) X is $Sm_Z^{\#}$

(ii) for each $\langle \varepsilon_n \rangle$ there is an $\langle \varepsilon_n \rangle$-fine γ-groupable cover of X

PRODUCTS

QUESTION (Schepers 1999) When is a product of Smz sets Smz? (CH example: $X \times X$ not Smz)

THM (Oz 2019) TFAE

(i) X is Smz$^{\#}$

(ii) $\check{H}^q(X \times Y) = 0$ whenever $\check{H}^q(Y) = 0$

(iii) $\check{H}^1(X \times E) = 0$ for all $E \in \mathcal{E}$

THM (Oz 2019)

$\longrightarrow$ X Smz$^{\#}$, Y Smz $\Rightarrow$ $X \times Y$ Smz

$\longrightarrow$ X Smz$^{\#}$, Y Smz$^{\#}$ $\Rightarrow$ $X \times Y$ Smz$^{\#}$

PRODUCTS

- $(\operatorname{cov} \mathcal{M} = \mathfrak{c})$ $\exists X \subseteq \mathbb{R}$ X Smz

  $X \times X$ not Smz
  $\Rightarrow X$ not Smz$^{\#}$

- $X, Y \subseteq G$ Smz$^{\#}$ $\Rightarrow$ $X + Y$ Smz$^{\#}$

- $X \subseteq G$ Smz$^{\#}$ $\Rightarrow \exists H \subseteq G$ subgroup
  $X \subseteq H$, H Smz$^{\#}$

- TFAE for $X \subseteq \mathbb{R}^2$
  - X is Smz$^{\#}$
  - all projections on lines are Smz$^{\#}$
  - two projections on lines are Smz$^{\#}$

CORAZZA'S MODEL

Corazza 1989 : forcing extension

- let R be the ground model reals
- $|R| = \omega_1$
- R is not meager in the extension
- $Sm_2(\mathbb{R}) = [\mathbb{R}]^{\leq \omega_1}$

PROP

- R is not $Sm_2^{\#}$

  $\vdash$ not meager $\dashv$

- $R \times X$ is Sm_2 for all $X \subseteq \mathbb{R}$ Sm_2

  $\vdash Y\ Sm_2 \Rightarrow |Y| \leq \omega_1 \Rightarrow |R \times X| \leq \omega_1 \Rightarrow$

  $R \times X\ Sm_2$ $\dashv$

QUESTION: Characterize sets S such that

$$X\ Sm_2 \Rightarrow S \times X\ Sm_2$$

GALVIN-LIKE GAME

$G^{\#}$

I	ε_0	ε_1	ε_2	.....
II	E_0	E_1	E_2	

Rules : $\longrightarrow$ $\varepsilon_n > 0$

$\longrightarrow$ $\operatorname{diam} E_n \leq \varepsilon_n$

II wins if $\{E_n\}$ is a γ-groupable cover

THM (Oz 2019) Let X be a separable metric space.

TFAE: (i) X is $\mathrm{Sm}_2^{\#}$

(ii) I does not have a winning strategy.

Note : X is not supposed σ-totally bounded.

PARTITION ARROW

$\longrightarrow$ Ω^{unif} ... uniform ω-covers

$\longrightarrow$ $\mathcal{O}^{\gamma\text{-gp}}$ ... γ-groupable covers

THM (Oz 2022+) TFAE

(i) X is $Sm_{\mathbb{Z}}^{\#}$

(ii) $\Omega^{unif} \longrightarrow (\mathcal{O}^{\gamma\text{-gp}})^2_2$ and X is σ-t.b.

(iii) $\Omega^{unif} \longrightarrow (\mathcal{O}^{\gamma\text{-gp}})^2_k$ $\forall k$ and X is σ-t.b.